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Factorizing twists and R-matrices for representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$

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Abstract

We calculate factorizing twists in evaluation representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$. From the factorizing twists we derive a representation-independent expression of the R-matrices of $U_q(\widehat{\mathfrak{sl}_2})$. Comparing with the corresponding quantities for the Yangian $Y(\mathfrak{sl}_2)$, it is shown that the $U_q(\widehat{\mathfrak{sl}_2})$ results can be obtained by 'replacing numbers by q-numbers'. Conversely, the limit $q \to 1$ exists in representations of $U_q(\widehat{\mathfrak{sl}_2})$ and both the factorizing twists and the R-matrices of the Yangian $Y(\mathfrak{sl}_2)$ are recovered in this limit.

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1. Introduction

The notion of twists and factorizing twists was introduced by Drinfel'd [1] in the context of quasi-Hopf algebras. They were applied very successfully to the Hopf algebras used in the framework of the algebraic Bethe ansatz for quantum integrable spin chains. Maillet and Sanchez de Santos found factorizing twists in this context for the fundamental evaluation representations of the Yangian $Y(\mathfrak{sl}_2)$ and of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ [2].

Such a factorizing twist F equips the quantum algebra with a new coproduct $\Delta_F = F \cdot \Delta \cdot F^{-1}$ which is cocommutative. In representations the twist corresponds to a change of basis on the tensor product, which is in general not induced by changes of basis of the tensor factors. In the new basis, the coproduct then becomes invariant under permutations of the tensor factors and is cocommutative.

The cocommutativity of the coproduct has provided a dramatic simplification of the algebraic Bethe ansatz. The application of factorizing twists to this ansatz has triggered a rapid development encompassing the solution of the inverse problem, the calculation of correlation functions (see e.g. [3–5]) and a simplification of the nesting procedure if algebras of higher rank are considered [6, 7].

Factorizing twists were constructed first for the fundamental evaluation representations of the Yangian $Y(\mathfrak{sl}_2)$ and of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ [2]. This was recently extended to arbitrary evaluation representations of the Yangian $Y(\mathfrak{sl}_2)$ [8, 9] and to the fundamental representation of the Yangian $Y(\mathfrak{sl}_n)$ for general n [6].

In this paper, we derive factorizing twists in finite-dimensional evaluation representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ in a direct calculation (q not a root of unity). It is possible to follow the approach presented in [8] and to generalize the results from the Yangian $Y(\mathfrak{sl}_2)$ almost literally to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$, replacing numbers by q-numbers. Conversely, the limit $q \to 1$ exists in representations of $U_q(\widehat{\mathfrak{sl}_2})$ both for the twist and for the R-matrix, and the corresponding Yangian results are recovered. This limit $q \to 1$ demonstrates explicitly the close relation of the two algebras at the representation level. One might have conjectured such a result from the fact that the quantum affine algebra essentially specializes to the Yangian if $q = 1$; see, e.g., the comments in [10, 11]. However, this relation has not been seen explicitly at the representation level before.

A similar correspondence between the Yangian $Y(\mathfrak{sl}_n)$ and the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_n})$ for higher n can be observed in their fundamental evaluation representations if the results of [6] are compared with the specialization of [7] to the trigonometric case.

In addition to the factorizing twists we determine the R-matrix for generic evaluation representations in its Gauss decomposition. In particular, no fusion is necessary to calculate the R-matrices of higher representations.

The key idea of our calculations is to choose a suitable presentation of the evaluation representations of $U_q(\widehat{\mathfrak{sl}_2})$ which exhibits close similarities with the Yangian $Y(\mathfrak{sl}_2)$, namely, compared with the Yangian, numbers are replaced by q-numbers. Then the machinery developed in [8] can be applied in a straightforward way. A remarkable result is that all intermediate steps in the calculation of the twists for the Yangian hold for q-numbers as well and thus generalize almost literally to $U_q(\widehat{\mathfrak{sl}_2})$. This reveals the deep similarity of $U_q(\widehat{\mathfrak{sl}_2})$ and $Y(\mathfrak{sl}_2)$ at the representation level.

At present the form of the factorizing twists on the abstract algebra $U_q(\widehat{\mathfrak{sl}_2})$ is not known. This is partly due to the fact that the twists were first discovered in representations rather than on the algebra, and it is not possible to just pull back the coproduct along the evaluation homomorphism $U_q(\widehat{\mathfrak{sl}_2}) \to U_q(\mathfrak{sl}_2)$ because this map does not preserve the co-algebra structure. Furthermore, one does not expect a factorizing twist as a proper element of $U_q(\widehat{\mathfrak{sl}_2}) \otimes U_q(\widehat{\mathfrak{sl}_2})$, but only as a formal expression which makes sense on a class of representations which is determined in section 3.3.

This paper is organized as follows. In section 2 we recall the basic properties of factorizing twists and introduce our notation for the study of evaluation representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$. The factorizing twists in a generic evaluation representation are then constructed in section 3. The calculations follow closely the lines of [8], and we keep the presentation of the results very brief. The precise conditions for the existence of the twists in finite-dimensional evaluation representations are given in section 3.3. In section 4 we construct R-matrices in evaluation representations of $U_q(\widehat{\mathfrak{sl}_2})$. Examples of factorizing twists are finally tabulated in section 5.

2. Preliminaries

2.1. Drinfel'd twists

First we recall the definition and key properties of factorizing twists. They are due to Drinfel'd [1, 10], where more details can be found. We write μ, η, Δ, ε, S for the product, unit, coproduct, co-unit and antipode of a Hopf algebra.

Definition 2.1. *A Hopf algebra $\mathcal{A}$ is called quasi-triangular if there exists an invertible element*

$R \in \mathcal{A} \otimes \mathcal{A}$, *called the universal R-matrix, which satisfies*

$$(\Delta \otimes \mathrm{id})(R) = R_{13} R_{23} \tag{2.1a}$$

$$(\mathrm{id} \otimes \Delta)(R) = R_{13} R_{12} \tag{2.1b}$$

$$\Delta^{\mathrm{op}}(a) = R \cdot \Delta(a) \cdot R^{-1} \tag{2.1c}$$

for all $a \in \mathcal{A}$. *The Hopf algebra* $\mathcal{A}$ *is called triangular if in addition*

$$R_{21} = R_{12}^{-1}. \tag{2.2}$$

Here R_{ij} *denote as usual the embeddings of R in the different factors of* $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$.

Definition 2.2. *Let* $\mathcal{A}$ *be a Hopf algebra. An invertible element* $F \in \mathcal{A}\otimes\mathcal{A}$ *is called a co-unital* 2*-cocycle or Drinfel'd twist if it satisfies*

$$(\varepsilon \otimes \mathrm{id})(F) = 1 \tag{2.3a}$$

$$(\mathrm{id} \otimes \varepsilon)(F) = 1 \tag{2.3b}$$

$$F_{12} \cdot (\Delta \otimes \mathrm{id})(F) = F_{23} \cdot (\mathrm{id} \otimes \Delta)(F). \tag{2.3c}$$

Theorem 2.3 (Drinfel'd). *Let* $\mathcal{A}$ *be a quasi-triangular Hopf algebra and* $F \in \mathcal{A} \otimes \mathcal{A}$ *be a co-unital* 2*-cocycle. Then the algebra of* $\mathcal{A}$ *together with the operations*

$$\Delta_F(a) := F \cdot \Delta(a) \cdot F^{-1} \tag{2.4a}$$

$$S_F(a) := u \cdot S(a) \cdot u^{-1} \qquad u := \mu(\mathrm{id} \otimes S)(F) \tag{2.4b}$$

$$R_F := F_{21} \cdot R \cdot F^{-1} \tag{2.4c}$$

and the old co-unit ε *forms a quasi-triangular Hopf algebra* $\mathcal{A}_F$.

The cocycle condition (2.3c) is required in order to make the twisted coproduct Δ_F coassociative so that one obtains a Hopf algebra $\mathcal{A}_F$ rather than just a quasi-Hopf algebra.

Definition 2.4. *Let* $\mathcal{A}$ *be a quasi-triangular Hopf algebra with a co-unital* 2*-cocycle* $F \in \mathcal{A} \otimes \mathcal{A}$. F *is called a factorizing twist if* $R_F = 1 \otimes 1$ *in theorem 2.3, i.e.*

$$R_{12} = F_{21}^{-1} \cdot F_{12}. \tag{2.5}$$

Remark 2.5.

(1) If a quasi-triangular Hopf algebra $\mathcal{A}$ admits a factorizing twist, the twisted coproduct Δ_F is cocommutative.

(2) In this case the Hopf algebra is triangular since its universal R-matrix satisfies

$$R_{21} = F_{12}^{-1} \cdot F_{21} = R_{12}^{-1}. \tag{2.6}$$

(3) The converse implication is true at least for finite-dimensional semi-simple Hopf algebras [12].

(4) If F is a factorizing twist, then the opposite coproduct can be made cocommutative using the twist F_{21} rather than F_{12}:

$$F_{21} \cdot \Delta^{\mathrm{op}}(a) \cdot F_{21}^{-1} = \Delta_F(a) = F_{12} \cdot \Delta(a) \cdot F_{12}^{-1}. \tag{2.7}$$

In the discussion of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ we face the difficulty that these algebras do not have a universal R-matrix as an element of $\mathcal{A} \otimes \mathcal{A}$, but rather a pseudo-universal R-matrix, i.e. a formal expression which gives rise to a corresponding quantity only on a certain class of representations (in this case on the irreducible representations). It is furthermore known [11] that the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ is pseudo-triangular.

We thus expect only a pseudo-factorizing twist, i.e. the twist should exist for a large class of representations, but not be an element of $\mathcal{A}\otimes\mathcal{A}$. In section 3.3 we state precise conditions for which evaluation representations our formal expressions give rise to well defined and invertible linear maps.

2.2. Notations and conventions

In this section we explain our notation for finite-dimensional evaluation representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$. The central idea is to employ a notation for $U_q(\widehat{\mathfrak{sl}_2})$ which shows sufficiently close similarities with the Yangian $Y(\mathfrak{sl}_2)$ in order to apply the methods of [8].

Similarly to the case of the Yangian $Y(\mathfrak{sl}_2)$ whose evaluation representations can be described using representations of the Lie algebra $\mathfrak{sl}_2$, it is possible to write the finite-dimensional evaluation representations of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ in terms of certain representations of the algebra $U_q(\mathfrak{sl}_2)$ (without hat) [13]. There are two versions of these algebras, one in terms of formal power series in $q-1$, the other with a special complex number q. In the latter case, the representations have essentially the same structure only if q is not a root of unity.

The algebra $U_q(\mathfrak{sl}_2)$ is defined [14], in terms of the generators E, F, K and K^{-1} and the relations

$$K\,K^{-1} = 1 = K^{-1}\,K \tag{2.8a}$$

$$K\,E\,K^{-1} = q^2\,E \tag{2.8b}$$

$$K\,F\,K^{-1} = q^{-2}\,F \tag{2.8c}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \tag{2.8d}$$

for $q \notin \{-1, 1\}$.

If q is not a root of unity, its finite-dimensional irreducible complex representations are given up to equivalence by the vector spaces $V_{\varepsilon,n} \cong \mathbb{C}^{n+1}$, $\varepsilon \in \{-1, 1\}$, $n \in \mathbb{N}_0$. $U_q(\mathfrak{sl}_2)$ acts on a basis $|0\rangle, \ldots, |n\rangle$ of $V_{\varepsilon,n}$ as follows:

$$E|m\rangle = \varepsilon[m]_q\,|m-1\rangle \tag{2.9a}$$

$$F|m\rangle = [n-m]_q\,|m+1\rangle \tag{2.9b}$$

$$K|m\rangle = \varepsilon\,q^{n-2m}\,|m\rangle. \tag{2.9c}$$

Here we have used the q-numbers

$$[k]_q := \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{2.10}$$

In order to describe the evaluation representations of $U_q(\widehat{\mathfrak{sl}_2})$, we need only the 'type 1' representations $V_{1,n}$ for which $\varepsilon = 1$. There is a Casimir element

$$C^{(2)} = \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2} + FE \tag{2.11}$$

whose eigenvalue on $V_{1,n}$ is given by

$$\frac{q^{n+1} + q^{-n-1}}{(q - q^{-1})^2}. \tag{2.12}$$

It is convenient to cast these representations in a form which looks more similar to the Lie algebra $\mathfrak{sl}_2$ in Cartan–Weyl form. Therefore we define for each representation $V_{1,n}$ an operator H by

$$H|m\rangle := \left(\frac{n}{2} - m\right)\,|m\rangle \tag{2.13}$$

i.e. $K = q^{2H}$ so that E, F and H satisfy the following identities on representations $V_{1,n}$:

$$[H, E] = E \tag{2.14a}$$

$$[H, F] = -F \tag{2.14b}$$

$$[E, F] = [2H]_q. \tag{2.14c}$$

In this formulation, the limit $q \to 1$ makes sense, and the generators E and F tend towards their $\mathfrak{sl}_2$ counterparts while H remains unchanged. The vector $|0\rangle$ then corresponds to the highest-weight vector with weight $\frac{n}{2}$ of $\mathfrak{sl}_2$. The point is here to illustrate the limit $q \to 1$ for given representations rather than analysing the representations of the algebra (2.14).

The finite-dimensional evaluation representations $V_\lambda(w)$ of $U_q(\widehat{\mathfrak{sl}_2})$ are characterized by a representation $V_{1,2\lambda}$ of $U_q(\mathfrak{sl}_2)$, $2\lambda \in \mathrm{N}_0$, q not a root of unity, and by a parameter $w \in \mathrm{C}$ [13]. Here we use the physicists' presentation of $U_q(\widehat{\mathfrak{sl}_2})$ in terms of generators $T^i_j(z)$ which satisfy the quantum Yang–Baxter equation in the form of the RTT-relations

$$R^{ij}_{k\ell}(v/z)T^k_p(v)T^\ell_q(z) = T^j_\ell(z)T^i_k(v)R^{k\ell}_{pq}(v/z) \tag{2.15}$$

with the trigonometric R-matrix

$$R(z) = \begin{pmatrix} 1 & & & \\ & \frac{z-z^{-1}}{zq-(zq)^{-1}} & \frac{q-q^{-1}}{zq-(zq)^{-1}} & \\ & \frac{q-q^{-1}}{zq-(zq)^{-1}} & \frac{z-z^{-1}}{zq-(zq)^{-1}} & \\ & & & 1 \end{pmatrix}. \tag{2.16}$$

The functionals $T^i_j(z)$ in (2.15) represent the generators of $U_q(\widehat{\mathfrak{sl}_2})$ in the evaluation representation $V_\lambda(w)$. They read in terms of the operators (2.14)

$$A(z) = T^1_1(z) = \frac{\frac{z}{w}q^{H+\frac{1}{2}} - \frac{w}{z}q^{-H-\frac{1}{2}}}{q - q^{-1}} \tag{2.17a}$$

$$B(z) = T^1_2(z) = F \tag{2.17b}$$

$$C(z) = T^2_1(z) = E \tag{2.17c}$$

$$D(z) = T^2_2(z) = \frac{\frac{z}{w}q^{-H+\frac{1}{2}} - \frac{w}{z}q^{H-\frac{1}{2}}}{q - q^{-1}}. \tag{2.17d}$$

Note that there are several versions of (2.16) and (2.17) in the literature which are related by changes of the basis that can depend on z. For a comparison with the Yangian, the above choice is most convenient.

Finally, we write $z = q^u$, $w = q^\delta$ and $q = \mathrm{e}^h$ so that the spectral parameter becomes additive, and the trigonometric nature of the R-matrix is apparent. We find the following expressions:

$$A(u) = T^1_1(z) = [u - \delta + H + \tfrac{1}{2}]_q \tag{2.18a}$$

$$B(u) = T^1_2(z) = F \tag{2.18b}$$

$$C(u) = T^2_1(z) = E \tag{2.18c}$$

$$D(u) = T^2_2(z) = [u - \delta - H + \tfrac{1}{2}]_q. \tag{2.18d}$$

In the following, we use this notation in terms of q-numbers in order to make the similarities with the Yangian obvious. With this notation, however, we always mean the corresponding expressions in terms of z, w and q as given in (2.17).

Observe that in (2.17) we have chosen a particular normalization. With this choice, the quantum determinant is given by

$$\mathrm{qdet}T(z) = \frac{\frac{z^2q}{w^2} + \frac{w^2}{z^2q}}{(q - q^{-1})^2} - C^{(2)} \tag{2.19}$$

where $C^{(2)}$ denotes the Casimir element of $U_q(\mathfrak{sl}_2)$. On evaluation of representations $V_\lambda(w)$ the above equation reads

$$\mathrm{qdet}T(z) = \frac{\frac{z^2q}{w^2} + \frac{w^2}{z^2q}}{(q-q^{-1})^2} - \frac{q^{2\lambda+1} + q^{-2\lambda-1}}{(q-q^{-1})^2} = [u-\delta+\lambda+1]_q\,[u-\delta-\lambda]_q \tag{2.20}$$

where $z = q^u$ and $w = q^\delta$. Therefore, strictly speaking, we treat $V_\lambda(w)$ as a representation of $U_q(\widehat{\mathfrak{gl}}_2)$.

For compatibility with [8, 9], we use the following coproduct (which might be more naturally called the opposite coproduct),

$$\Delta A(u) = A(u) \otimes A(u) + C(u) \otimes B(u) \tag{2.21a}$$
$$\Delta C(u) = A(u) \otimes C(u) + C(u) \otimes D(u) \tag{2.21b}$$
$$\Delta B(u) = B(u) \otimes A(u) + D(u) \otimes B(u) \tag{2.21c}$$
$$\Delta D(u) = B(u) \otimes C(u) + D(u) \otimes D(u). \tag{2.21d}$$

This coproduct acts on $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ as

$$\Delta A(u)|\ell,k\rangle = a_\ell^{(1)}(u)\,a_k^{(2)}(u)|\ell,k\rangle + [\ell]_q[2\lambda_2-k]_q|\ell-1,k+1\rangle \tag{2.22a}$$
$$\Delta B(u)|\ell,k\rangle = a_k^{(2)}(u)\,[2\lambda_1-\ell]_q|\ell+1,k\rangle + d_\ell^{(1)}(u)\,[2\lambda_2-k]_q|\ell,k+1\rangle \tag{2.22b}$$
$$\Delta C(u)|\ell,k\rangle = a_\ell^{(1)}(u)\,[k]_q|\ell,k-1\rangle + d_k^{(2)}(u)\,[\ell]_q|\ell-1,k\rangle \tag{2.22c}$$
$$\Delta D(u)|\ell,k\rangle = [2\lambda_1-\ell]_q[k]_q|\ell+1,k-1\rangle + d_\ell^{(1)}(u)\,d_k^{(2)}(u)\,|\ell,k\rangle \tag{2.22d}$$

where we write $|\ell,k\rangle := |\ell\rangle \otimes |k\rangle$, $w_j = q^{\delta_j}$ and

$$a_k^{(j)}(u) := [u-\delta_j+\lambda_j-k+\tfrac{1}{2}]_q \tag{2.23a}$$
$$d_k^{(j)}(u) := [u-\delta_j-\lambda_j+k+\tfrac{1}{2}]_q. \tag{2.23b}$$

2.3. Irreducibility of evaluation representations of $U_q(\widehat{\mathfrak{sl}}_2)$

The following statements concerning evaluation representations of the quantum affine algebra go back to the work of Tarasov [15] and can be found, for example, in [13].

Theorem 2.6. *Each finite-dimensional irreducible type* $(1,1)$ *representation of the quantum affine algebra* $U_q(\widehat{\mathfrak{sl}}_2)$ *over* C *is isomorphic to a tensor product of evaluation representations. Two such tensor products describe isomorphic representations if and only if they are related by a permutation of the tensor factors.*

Theorem 2.7. *The tensor product of finite-dimensional evaluation representations* $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ *of* $U_q(\widehat{\mathfrak{sl}}_2)$ *is reducible if and only if*

$$\frac{w_1^2}{w_2^2} = q^{\pm 2(\lambda_1+\lambda_2-j+1)} \tag{2.24}$$

for an integer j *satisfying* $1 \leqslant j \leqslant \min\{2\lambda_1, 2\lambda_2\}$. *In this case the representation is neither completely reducible nor isomorphic to* $V_{\lambda_2}(w_2) \otimes V_{\lambda_1}(w_1)$.

Note that we have adapted the above result to our notation, which uses a basis of Cartan–Weyl type.

3. Construction of factorizing twists

The strategy to construct the factorizing twists is the following. In each evaluation representation, the twist F_{12}^{-1} is given by a lower triangular matrix. It is calculated in a form decomposed into a diagonal matrix multiplied by a lower triangular matrix which has only entries '1' on its diagonal. The triangular part of the twist is the change of basis operator on the tensor product $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ which diagonalizes $\Delta D(u)$. The prefactors of the eigenvectors of $\Delta D(u)$ form the diagonal part of the twist. They are fixed in a second step by the requirement that the twisted coproduct has to be cocommutative.

3.1. The triangular part

It is convenient to make use of the following abbreviations, cf the corresponding definitions for the Yangian in equations (3.10c) and (3.16c) of [8]:

$$g(x) := [\delta_1 - \delta_2 + \lambda_1 - \lambda_2 + x]_q \tag{3.1a}$$

$$\tilde{g}(x) := [\delta_2 - \delta_1 + \lambda_1 - \lambda_2 + x]_q. \tag{3.1b}$$

Furthermore we define the q-factorial $[n]_q! := [n]_q\,[n-1]_q \cdots [1]_q$; $[0]_q! := 1$ for $n \in \mathrm{N}$.

Using the same method as in section 3 of [8], we can diagonalize $\Delta D(u)$ on each representation $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$. The remarkable result is that the structure of the solutions does not change compared with the Yangian $Y(\mathfrak{sl}_2)$. The only difference is the appearance of the q-version of $g(x)$ (3.1a).

Lemma 3.1. *The eigenvectors of $\Delta D(u)$ on $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ are independent of u. They are given by*

$$v_{\ell k} = q_{\ell k} \sum_{n=0}^{\min\{k, 2\lambda_1 - \ell\}} \frac{(-1)^n}{[n]_q!} \left(\prod_{j=1}^{n} \frac{[2\lambda_1 - \ell - j + 1]_q [k - j + 1]_q}{g(k - \ell - j)} \right) |\ell + n, k - n\rangle \tag{3.2}$$

and correspond to the eigenvalue $d_\ell^{(1)}(u) \cdot d_k^{(2)}(u)$. Here the $q_{\ell k}$ are arbitrary factors. The inverse transformation is given by

$$|\ell, k\rangle = \sum_{n=0}^{\min\{k, 2\lambda_1 - \ell\}} \frac{1}{[n]_q!} \left(\prod_{j=1}^{n} \frac{[2\lambda_1 - \ell - j + 1]_q [k - j + 1]_q}{g((k-n) - (\ell + n) + j)} \right) \frac{v_{\ell+n, k-n}}{q_{\ell+n, k-n}}. \tag{3.3}$$

This change of basis determines the triangular part of the factorizing twist:

Proposition 3.2. *The expression*

$$F_{12}^{-1} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_q!} F^n \otimes E^n \prod_{j=1}^{n} [\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H - j]_q{}^{-1} \right) Q_{12}^{-1} \tag{3.4}$$

specializes to the change of basis (3.2) on all representations $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$. The expression

$$F_{12} = Q_{12} \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \left(\prod_{j=1}^{n} [\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H + j]_q{}^{-1} \right) F^n \otimes E^n \tag{3.5}$$

specializes to the inverse change of basis (3.3). Here

$$Q_{12}^{-1} |\ell, k\rangle = q_{\ell k} |\ell, k\rangle \tag{3.6}$$

denotes the diagonal part of the twist.

In addition to this twist, which diagonalizes $\Delta D(u)$ on evaluation representations, it is possible to construct another twist from the requirement that it diagonalizes $\Delta A(u)$.

Lemma 3.3. *The eigenvectors of $\Delta A(u)$ on $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ are independent of u. They are given by*

$$\tilde{v}_{\ell k} = \tilde{q}_{\ell k} \sum_{n=0}^{\min\{2\lambda_2 - k, \ell\}} \frac{1}{[n]_q!} \left(\prod_{j=1}^{n} \frac{[2\lambda_2 - k - j + 1]_q [\ell - j + 1]_q}{\tilde{g}(k - \ell + j)} \right) |\ell - n, k + n\rangle \tag{3.7}$$

and correspond to the eigenvalues $a_\ell^{(1)}(u) \cdot a_k^{(2)}(u)$. Here the $\tilde{q}_{\ell k}$ are arbitrary factors. The inverse transformation is given by

$$|\ell, k\rangle = \sum_{n=0}^{\min\{2\lambda_2 - k, \ell\}} \frac{(-1)^n}{[n]_q!} \left(\prod_{j=1}^{n} \frac{[2\lambda_2 - k - j + 1]_q [\ell - j + 1]_q}{\tilde{g}((k + n) - (\ell - n) - j)} \right) \frac{\tilde{v}_{\ell - n, k + n}}{\tilde{q}_{\ell - n, k + n}}. \tag{3.8}$$

This change of basis determines the triangular part of another factorizing twist:

Proposition 3.4. *The expressions*

$$\tilde{F}_{12}^{-1} = \left(\sum_{n=0}^{\infty} \frac{1}{[n]_q!} E^n \otimes F^n \prod_{j=1}^{n} [\delta_2 - \delta_1 + H \otimes 1 - 1 \otimes H + j]_q{}^{-1} \right) \tilde{Q}_{12}^{-1} \tag{3.9a}$$

$$\tilde{F}_{12} = \tilde{Q}_{12} \sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_q!} \left(\prod_{j=1}^{n} [\delta_2 - \delta_1 + H \otimes 1 - 1 \otimes H - j]_q{}^{-1} \right) E^n \otimes F^n \tag{3.9b}$$

specialize to the change of basis (3.7) and its inverse transformation (3.8) on all representations $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$. Here

$$\tilde{Q}_{12}^{-1} |\ell, k\rangle = \tilde{q}_{\ell k} |\ell, k\rangle \tag{3.10}$$

denotes the diagonal part of the twist.

3.2. The diagonal part

At this point we know two different operators F_{12}^{-1} and $\tilde{F}_{12}^{-1}$ which provide a change of basis on the tensor products of finite-dimensional evaluation representations $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$. The first one diagonalizes $\Delta D(u)$ and the second $\Delta A(u)$. Their diagonal parts Q_{12}^{-1} and $\tilde{Q}_{12}^{-1}$ are as yet unspecified.

Employing the same method as in section 3 of [8], it can be shown that a suitable choice of the diagonal part Q_{12}^{-1} leads to a cocommutative coproduct $\Delta_F = F_{12} \cdot \Delta \cdot F_{12}^{-1}$. The required conditions on the diagonal part are stated in the following propositions.

Proposition 3.5. *If the coefficients $q_{\ell k}$ of Q_{12}^{-1} satisfy the recursion relations*

$$\frac{q_{\ell+1,k}}{q_{\ell k}} = \frac{g(-\ell - 1)}{g(k - \ell - 1)} \qquad \frac{q_{\ell,k+1}}{q_{\ell k}} = \frac{g(k - \ell)}{g(-2\lambda_1 + k)} \tag{3.11}$$

for $0 \leqslant \ell \leqslant 2\lambda_1$ and $0 \leqslant k \leqslant 2\lambda_2$, then the twisted coproduct on $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ is cocommutative. It has the form

$$F_{12} \cdot \Delta D(u) \cdot F_{12}^{-1} = D(u) \otimes D(u) \tag{3.12a}$$

$$\begin{aligned} F_{12} \cdot \Delta B(u) \cdot F_{12}^{-1} = {} & B(u) \otimes D(u) \frac{[\delta_1 - \delta_2 + H \otimes 1 + \lambda_2]_q}{[\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H]_q} \\ & + D(u) \otimes B(u) \frac{[\delta_1 - \delta_2 - \lambda_1 - 1 \otimes H]_q}{[\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H]_q} \end{aligned} \tag{3.12b}$$

$$F_{12} \cdot \Delta C(u) \cdot F_{12}^{-1} = C(u) \otimes D(u) \, \frac{[\delta_1 - \delta_2 + H \otimes 1 - \lambda_2]_q}{[\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H]_q}$$
$$+ D(u) \otimes C(u) \, \frac{[\delta_1 - \delta_2 + \lambda_1 - 1 \otimes H]_q}{[\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H]_q}. \tag{3.12c}$$

Proof. The proof is almost literally the same as for proposition 4.1 of [8]. The assertions reduce to the following identities for q-numbers for $0 \leqslant \ell \leqslant 2\lambda_1$, $0 \leqslant k \leqslant 2\lambda_2$, $0 \leqslant i \leqslant k$:

$$a_{k-i}^{(2)}(u) - d_{\ell+i+1}^{(1)}(u) \, \frac{[k-i]_q [2\lambda_2 - k + i + 1]_q}{[i+1]_q \, g(k - \ell - i - 1)}$$
$$= d_k^{(2)}(u) \, \frac{g(2\lambda_2 - \ell) \, g(-\ell - 1)}{g(k-\ell) \, g(k - \ell - i - 1)} - d_\ell^{(1)}(u) \, \frac{[k+1]_q [2\lambda_2 - k]_q}{[i+1]_q \, g(k-\ell)} \tag{3.13a}$$

and for $1 \leqslant i \leqslant k$,

$$d_{k-i}^{(2)}(u) \, [\ell + i]_q [2\lambda_1 - \ell - i + 1]_q - a_{\ell+i-1}^{(1)}(u) \, [i]_q \, g(k - \ell - i)$$
$$= d_k^{(2)}(u) \, [\ell]_q [2\lambda_1 - \ell + 1]_q \, \frac{g(k - \ell - i)}{g(k - \ell)} - d_\ell^{(1)}(u) \, [i]_q \, \frac{g(k) \, g(-2\lambda_1 + k - 1)}{g(k - \ell)}. \tag{3.13b}$$

It is a remarkable and non-trivial fact that these identities, which are known to hold without the $[\cdot]_q$-brackets, are still valid in this form, which involves the q-numbers. □

Remark 3.6. We can use the quantum determinant (2.20) in order to prove that $F_{12} \cdot \Delta A(u) \cdot F_{12}^{-1}$ is also cocommutative.

Lemma 3.7. *The coefficients $q_{\ell k}$, given by*

$$q_{\ell k} = \prod_{j=0}^{k-1} \frac{[\delta_1 - \delta_2 + \lambda_1 - \lambda_2 - \ell + j]_q}{[\delta_1 - \delta_2 - \lambda_1 - \lambda_2 + j]_q} \tag{3.14}$$

satisfy the recursion relations (3.11).

In section 5.3 of [8], a second choice of the diagonal part gave rise to a coproduct with a different ordering of operators. This construction has also a correspondence in $U_q(\widehat{\mathfrak{sl}}_2)$:

Proposition 3.8. *Let $\hat{F}_{12}^{-1}$ denote the expression (3.4) with a diagonal part $\hat{Q}_{12}^{-1}$ rather than Q_{12}^{-1},*

$$\hat{F}_{12}^{-1} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{[n]_q!} F^n \otimes E^n \prod_{j=1}^{n} [\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H - j]_q{}^{-1} \right) \hat{Q}_{12}^{-1} \tag{3.15a}$$

$$\hat{Q}_{12}^{-1} |\ell, k\rangle = \hat{q}_{\ell k} |\ell, k\rangle. \tag{3.15b}$$

If the coefficients $\hat{q}_{\ell k}$ satisfy

$$\frac{\hat{q}_{\ell+1,k}}{\hat{q}_{\ell k}} = \frac{g(2\lambda_2 - \ell)}{g(k - \ell)} \qquad \frac{\hat{q}_{\ell,k+1}}{\hat{q}_{\ell k}} = \frac{g(k - \ell + 1)}{g(k+1)} \tag{3.16}$$

for $0 \leqslant \ell \leqslant 2\lambda_1$, $0 \leqslant k \leqslant 2\lambda_2$, then the coproduct twisted by $\hat{F}$ is also cocommutative. It is given by

$$\hat{F}_{12} \cdot \Delta D(u) \cdot \hat{F}_{12}^{-1} = D(u) \otimes D(u) \tag{3.17a}$$

$$\hat{F}_{12} \cdot \Delta B(u) \cdot \hat{F}_{12}^{-1} = \frac{[\delta_1 - \delta_2 + H \otimes 1 - \lambda_2]_q}{[\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H]_q} \, B(u) \otimes D(u)$$

$$+\frac{[\delta_1-\delta_2+\lambda_1-1\otimes H]_q}{[\delta_1-\delta_2+H\otimes 1-1\otimes H]_q}\,D(u)\otimes B(u) \tag{3.17b}$$

$$\hat{F}_{12}\cdot\Delta C(u)\cdot\hat{F}_{12}^{-1}=\frac{[\delta_1-\delta_2+H\otimes 1+\lambda_2]_q}{[\delta_1-\delta_2+H\otimes 1-1\otimes H]_q}\,C(u)\otimes D(u)$$
$$+\frac{[\delta_1-\delta_2-\lambda_1-1\otimes H]_q}{[\delta_1-\delta_2+H\otimes 1-1\otimes H]_q}\,D(u)\otimes C(u). \tag{3.17c}$$

The proof of this proposition reduces to the same identities (3.13).

Lemma 3.9. *The coefficients $\hat{q}_{\ell k}$, given by*

$$\hat{q}_{\ell k}=\prod_{j=1}^{\ell}\frac{[\delta_1-\delta_2+\lambda_1+\lambda_2-\ell+j]_q}{[\delta_1-\delta_2+\lambda_1-\lambda_2+k-\ell+j]_q} \tag{3.18}$$

satisfy the recursion relations (3.16).

Finally, analogous constructions are available for the twist $\tilde{F}_{12}^{-1}$ which diagonalizes $\Delta A(u)$ rather than $\Delta D(u)$:

Proposition 3.10. *If the coefficients $\tilde{q}_{\ell k}$ of the diagonal part $\tilde{Q}_{12}^{-1}$ of the transformation $\tilde{F}_{12}^{-1}$ in (3.9a) satisfy*

$$\frac{\tilde{q}_{\ell+1,k}}{\tilde{q}_{\ell k}}=\frac{\tilde{g}(k-\ell)}{\tilde{g}(2\lambda_2-\ell)}\qquad\frac{\tilde{q}_{\ell k}}{\tilde{q}_{\ell,k-1}}=\frac{\tilde{g}(k)}{\tilde{g}(k-\ell)} \tag{3.19}$$

for $0\leqslant\ell\leqslant 2\lambda_1$ and $0\leqslant k\leqslant 2\lambda_2$, then the twisted coproduct is cocommutative and given by

$$\tilde{F}_{12}\cdot\Delta A(u)\cdot\tilde{F}_{12}^{-1}=A(u)\otimes A(u) \tag{3.20a}$$

$$\tilde{F}_{12}\cdot\Delta B(u)\cdot\tilde{F}_{12}^{-1}=B(u)\otimes A(u)\,\frac{[\delta_2-\delta_1+H\otimes 1+\lambda_2]_q}{[\delta_2-\delta_1+H\otimes 1-1\otimes H]_q}$$
$$+A(u)\otimes B(u)\,\frac{[\delta_2-\delta_1-\lambda_1-1\otimes H]_q}{[\delta_2-\delta_1+H\otimes 1-1\otimes H]_q} \tag{3.20b}$$

$$\tilde{F}_{12}\cdot\Delta C(u)\cdot\tilde{F}_{12}^{-1}=C(u)\otimes A(u)\,\frac{[\delta_2-\delta_1+H\otimes 1-\lambda_2]_q}{[\delta_2-\delta_1+H\otimes 1-1\otimes H]_q}$$
$$+A(u)\otimes C(u)\,\frac{[\delta_2-\delta_1+\lambda_1-1\otimes H]_q}{[\delta_2-\delta_1+H\otimes 1-1\otimes H]_q}. \tag{3.20c}$$

Proof. The proof of this proposition is similar to that of proposition 3.5. It reduces to the following true identities for q-numbers where $0\leqslant\ell\leqslant 2\lambda_1$, $0\leqslant k\leqslant 2\lambda_2$ and $0\leqslant i\leqslant\ell$,

$$a_{k+i}^{(2)}(u)\,[\ell-i+1]_q[2\lambda_1-\ell+i]_q+d_{\ell-i+1}^{(1)}(u)\,[i]_q\,\tilde{g}(k-\ell+i)$$
$$=a_k^{(2)}(u)\,[\ell+1]_q[2\lambda_1-\ell]_q\,\frac{\tilde{g}(k-\ell+i)}{\tilde{g}(k-\ell)}+a_\ell^{(1)}(u)\,[i]_q\,\frac{\tilde{g}(-2\lambda_1+k)\,\tilde{g}(k+1)}{\tilde{g}(k-\ell)} \tag{3.21a}$$

and for $1\leqslant i\leqslant\ell$,

$$a_{\ell-i}^{(1)}(u)\,[k+i]_q[2\lambda_2-k-i+1]_q+d_{k+i-1}^{(2)}(u)\,[i]_q\tilde{g}(k-\ell+i)$$
$$=a_k^{(2)}(u)\,[i]_q\,\frac{\tilde{g}(-\ell)\,\tilde{g}(2\lambda_2-\ell+1)}{\tilde{g}(k-\ell)}+a_\ell^{(1)}(u)\,[k]_q[2\lambda_2-k+1]_q\,\frac{\tilde{g}(k-\ell+1)}{\tilde{g}(k-\ell)}. \tag{3.21b}$$

□

Lemma 3.11. *The coefficients $\tilde{q}_{\ell k}$,*

$$\tilde{q}_{\ell k} = \prod_{j=1}^{\ell} \frac{[\delta_2 - \delta_1 + \lambda_1 - \lambda_2 + k - \ell + j]_q}{[\delta_2 - \delta_1 + \lambda_1 + \lambda_2 - \ell + j]_q} \tag{3.22}$$

satisfy the recursion formulas (3.19).

Again, there is an alternative choice for the diagonal part of the twist:

Proposition 3.12. *Let $\hat{\tilde{F}}_{12}^{-1}$ denote the same operator as $\tilde{F}_{12}^{-1}$ in (3.9a), but with a diagonal part given by $\hat{\tilde{Q}}_{12}^{-1}$,*

$$\hat{\tilde{F}}_{12}^{-1} = \left(\sum_{n=0}^{\infty} \frac{1}{[n]_q!} E^n \otimes F^n \prod_{j=1}^{n} [\delta_2 - \delta_1 + H \otimes 1 - 1 \otimes H + j]_q{}^{-1} \right) \hat{\tilde{Q}}_{12}^{-1} \tag{3.23a}$$

$$\hat{\tilde{Q}}_{12}^{-1} |\ell, k\rangle = \hat{\tilde{q}}_{\ell k} |\ell, k\rangle. \tag{3.23b}$$

If the coefficients $\hat{\tilde{q}}_{\ell k}$ satisfy

$$\begin{aligned} \frac{\hat{\tilde{q}}_{\ell+1,k}}{\hat{\tilde{q}}_{\ell k}} &= \frac{\tilde{g}(k - \ell - 1)}{\tilde{g}(-\ell - 1)} \\ \frac{\hat{\tilde{q}}_{\ell k}}{\hat{\tilde{q}}_{\ell,k-1}} &= \frac{\tilde{g}(-2\lambda_1 + k - 1)}{\tilde{g}(k - \ell - 1)} \end{aligned} \tag{3.24}$$

for $0 \leqslant \ell \leqslant 2\lambda_1$, $0 \leqslant k \leqslant 2\lambda_2$, then the coproduct twisted by $\hat{\tilde{F}}$ is cocommutative. It is of the form

$$\hat{\tilde{F}}_{12} \cdot \Delta A(u) \cdot \hat{\tilde{F}}_{12}^{-1} = A(u) \otimes A(u) \tag{3.25a}$$

$$\begin{aligned} \hat{\tilde{F}}_{12} \cdot \Delta B(u) \cdot \hat{\tilde{F}}_{12}^{-1} &= \frac{[\delta_2 - \delta_1 + H \otimes 1 - \lambda_2]_q}{[\delta_2 - \delta_1 + H \otimes 1 - 1 \otimes H]_q} B(u) \otimes A(u) \\ &\quad + \frac{[\delta_2 - \delta_1 + \lambda_1 - 1 \otimes H]_q}{[\delta_2 - \delta_1 + H \otimes 1 - 1 \otimes H]_q} A(u) \otimes B(u) \end{aligned} \tag{3.25b}$$

$$\begin{aligned} \hat{\tilde{F}}_{12} \cdot \Delta C(u) \cdot \hat{\tilde{F}}_{12}^{-1} &= \frac{[\delta_2 - \delta_1 + H \otimes 1 + \lambda_2]_q}{[\delta_2 - \delta_1 + H \otimes 1 - 1 \otimes H]_q} C(u) \otimes A(u) \\ &\quad + \frac{[\delta_2 - \delta_1 - \lambda_1 - 1 \otimes H]_q}{[\delta_2 - \delta_1 + H \otimes 1 - 1 \otimes H]_q} A(u) \otimes C(u). \end{aligned} \tag{3.25c}$$

The proof is again similar and reduces to the same identities (3.21).

Lemma 3.13. *The coefficients $\hat{\tilde{q}}_{\ell k}$, given by*

$$\hat{\tilde{q}}_{\ell k} = \prod_{j=0}^{k-1} \frac{[\delta_2 - \delta_1 - \lambda_1 - \lambda_2 + j]_q}{[\delta_2 - \delta_1 + \lambda_1 - \lambda_2 - \ell + j]_q} \tag{3.26}$$

satisfy the recursion relations (3.24).

For the Yangian it was possible [8] to express the diagonal parts of the various twists as quotients of gamma functions. The finite products like (3.14), (3.18), (3.22) and (3.26) then appear in the specialization on evaluation representations when the arguments of the gamma functions of numerator and denominator differ precisely by an integer number.

Here a similar construction is available which makes use of a q-deformed gamma function. There are several versions of such functions defined in the literature. We need here the 'modified q-gamma function' $\Gamma_q(t)$ (which is called $\hat{\Gamma}_q$ in [16]). It is characterized by the property

$$\frac{\Gamma_q(t+1)}{\Gamma_q(t)} = [t]_q = \frac{q^t - q^{-t}}{q - q^{-1}}. \tag{3.27}$$

Proposition 3.14. *On each weight vector $|\ell, k\rangle$ of the representation $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$, the expressions*

$$Q_{12}^{-1} = \frac{\Gamma_q(\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H)}{\Gamma_q(\delta_1 - \delta_2 + H \otimes 1 - \lambda_2)} \frac{\Gamma_q(\delta_1 - \delta_2 - \lambda_1 - \lambda_2)}{\Gamma_q(\delta_1 - \delta_2 - \lambda_1 - 1 \otimes H)} \tag{3.28a}$$

$$\hat{Q}_{12}^{-1} = \frac{\Gamma_q(\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H + 1)}{\Gamma_q(\delta_1 - \delta_2 + H \otimes 1 + \lambda_2 + 1)} \frac{\Gamma_q(\delta_1 - \delta_2 + \lambda_1 + \lambda_2 + 1)}{\Gamma_q(\delta_1 - \delta_2 + \lambda_1 - 1 \otimes H + 1)} \tag{3.28b}$$

have the eigenvalues $q_{\ell k}$ and $\hat{q}_{\ell k}$ given in (3.14) and (3.18). Furthermore we find

$$\hat{Q}_{21} = Q_{12}^{-1} \tag{3.29a}$$

$$\hat{Q}_{12}^{-1} = Q_{21} \tag{3.29b}$$

$$\tilde{Q}_{12}^{-1} = Q_{21}^{-1}|_{\delta_1 \leftrightarrow \delta_2} \tag{3.29c}$$

$$\hat{\tilde{Q}}_{12}^{-1} = Q_{12}|_{\delta_1 \leftrightarrow \delta_2}. \tag{3.29d}$$

Remark 3.15. Observe that whenever the expressions for the factorizing twists F_{12}^{-1}, Q_{12}^{-1} etc are written in a particular finite-dimensional evaluation representation, the limit $q \to 1$ is well defined. In this limit, the operators E and F of $U_q(\mathfrak{sl}_2)$ tend towards their $\mathfrak{sl}_2$ counterparts. On weight vectors, the quotients of q-gamma functions reduce to finite products like (3.14). Each factor of these products has a well defined $q \to 1$ limit in which q-numbers tend towards ordinary numbers.

In all cases, the corresponding results for the Yangian $Y(\mathfrak{sl}_2)$ are recovered, as a comparison with [8] reveals. The Yangian results appear in the form where $\eta = 1$ in the notation of [8].

Example matrices F_{12}^{-1} in finite-dimensional evaluation representations of low dimension are tabulated in section 5.

3.3. Existence of the twist in representations

Finally, we study for which finite-dimensional evaluation representations $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ the twist and its inverse exist. We restrict ourselves to the twist F_{12}^{-1} and its inverse F_{12}, cf (3.4) and (3.5).

Therefore, we have to write the coefficients of Q_{12}^{-1} and Q_{12} in a form in which numerator and denominator have no common factor. Furthermore we have to take into account that the numerator of the diagonal part might cancel factors from the denominator of the triangular part. The structure of the expressions is the same as for the Yangian $Y(\mathfrak{sl}_2)$ in section 5.4 of [8]:

Proposition 3.16. *The expression F_{12}^{-1}, see equation (3.4), is well defined in the representation $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ if and only if*

$$\frac{w_1^2}{w_2^2} \neq q^{2(\lambda_1+\lambda_2-j+1)} \tag{3.30}$$

for all integers j in the range $1 \leqslant j \leqslant \min\{2\lambda_1, 2\lambda_2\}$.

Proof. From an analogous calculation as for the Yangian (section 5.4 of [8]) we obtain a well defined F_{12}^{-1} if and only if

$$[\delta_1 - \delta_2 - \lambda_1 - \lambda_2 + j - 1]_q \neq 0 \tag{3.31}$$

for all integers j in the range $1 \leqslant j \leqslant \min\{2\lambda_1, 2\lambda_2\}$. We recall that $w_j = q^{\delta_j}$, and the assertion follows. □

Remark 3.17. According to theorem 2.7, F_{12}^{-1} thus exists for all finite-dimensional irreducible evaluation representations of $U_q(\widehat{\mathfrak{sl}_2})$.

Proposition 3.18. *The expression F_{12} in (3.5) is well defined in the representation $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ if and only if*

$$\frac{w_1^2}{w_2^2} \neq q^{2(\lambda_1+\lambda_2-j+1)} \tag{3.32}$$

for all integers j in the range $2 \leqslant j \leqslant 2\lambda_1 + 2\lambda_2$.

Remark 3.19. There exist irreducible representations $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$ of $U_q(\widehat{\mathfrak{sl}_2})$ for which F_{12} is not well defined. The situation is essentially the same as for the Yangian $Y(\mathfrak{sl}_2)$.

4. R-matrices

From the factorizing twist which was determined in a representation-independent fashion, it is possible to derive a representation-independent R-matrix. This R-matrix appears automatically in a canonical form, being Gauss decomposed into an upper triangular multiplied by a diagonal multiplied by a lower triangular part.

We obtain the following expression for the R-matrix on $V_{\lambda_1}(w_1) \otimes V_{\lambda_2}(w_2)$:

$$R_{12} = F_{21}^{-1} \cdot F_{12} = R_+ \cdot R_0 \cdot R_-. \tag{4.1}$$

Here the upper triangular part R_+ is the triangular part of F_{21}^{-1} and can be obtained from (3.4). The diagonal part is the product $R_0 = Q_{21}^{-1} \cdot Q_{12}$, cf (3.28), and the lower triangular part R_- can be read off from (3.5),

$$R_+ = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} E^n \otimes F^n \prod_{j=1}^{n} [\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H + j]_q^{-1} \tag{4.2a}$$

$$\begin{aligned} R_0 = {} & \frac{\Gamma_q(\delta_1 - \delta_2 + H \otimes 1 + \lambda_2 + 1)}{\Gamma_q(\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H + 1)} \cdot \frac{\Gamma_q(\delta_1 - \delta_2 + \lambda_1 - 1 \otimes H + 1)}{\Gamma_q(\delta_1 - \delta_2 + \lambda_1 + \lambda_2 + 1)} \\ & \times \frac{\Gamma_q(\delta_1 - \delta_2 + H \otimes 1 - \lambda_2)}{\Gamma_q(\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H)} \cdot \frac{\Gamma_q(\delta_1 - \delta_2 - \lambda_1 - 1 \otimes H)}{\Gamma_q(\delta_1 - \delta_2 - \lambda_1 - \lambda_2)} \end{aligned} \tag{4.2b}$$

$$R_- = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \left(\prod_{j=1}^{n} [\delta_1 - \delta_2 + H \otimes 1 - 1 \otimes H + j]_q^{-1} \right) F^n \otimes E^n. \tag{4.2c}$$

The alternative twist $\hat{F}_{12}$ in proposition 3.8 factorizes the same R-matrix,

$$\hat{F}_{21}^{-1} \cdot \hat{F}_{12} = R_{12} \tag{4.3}$$

because the two factors of the diagonal part are just exchanged, cf (3.29). The twist $\tilde{F}_{12}^{-1}$ which diagonalizes $\Delta A(u)$, cf (3.9a), is related to F_{12}^{-1} by

$$\tilde{F}_{12}^{-1} = F_{21}^{-1}|_{\delta_1 \leftrightarrow \delta_2}. \tag{4.4}$$

It thus factorizes the opposite R-matrix, but with negative spectral parameter,

$$\tilde{F}_{21}^{-1} \cdot \tilde{F}_{12} = R_{21}|_{\delta_1 \leftrightarrow \delta_2}. \tag{4.5}$$

From the calculation of the R-matrix via the factorizing twists, only the property of almost-cocommutativity (2.1c) is guaranteed by construction. However, this requirement has already determined the R-matrix up to a constant representation-dependent factor. Recall that our R is normalized such that $R|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle$. Therefore it differs from an R matrix derived via the (pseudo-) universal R-matrix by a multiplicative factor which is the character of the R-matrix. This situation is again completely analogous to the Yangian.

5. Examples

In this section, we tabulate factorizing twists for some finite-dimensional evaluation representations of low dimension. We list the matrices of the twists F_{12}^{-1} in a weight basis. The other twists and the R-matrices can then be calculated from F_{12}^{-1}. The ordering of the basis vectors of $V_{\lambda_1}(\delta_1) \otimes V_{\lambda_2}(\delta_2)$ is $|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, \ldots$.

Recall that due to the similarity of $U_q(\widehat{\mathfrak{sl}_2})$ with the Yangian $Y(\mathfrak{sl}_2)$, one obtains twists for the Yangian if all q-brackets in the following formulas are ignored.

$V_{\frac{1}{2}}(\delta_1) \otimes V_{\frac{1}{2}}(\delta_2)$:

$$F_{12}^{-1} = \begin{pmatrix} 1 & & & \\ & \frac{[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-1]_q} & & \\ & -\frac{1}{[\delta_1-\delta_2-1]_q} & 1 & \\ & & & 1 \end{pmatrix}. \tag{5.1}$$

$V_{\frac{1}{2}}(\delta_1) \otimes V_1(\delta_2)$:

$$F_{12}^{-1} = \begin{pmatrix} 1 & & & & & \\ & \frac{[\delta_1-\delta_2-\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{3}{2}]_q} & & & & \\ & 0 & \frac{[\delta_1-\delta_2+\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{3}{2}]_q} & & & \\ & -\frac{1}{[\delta_1-\delta_2-\frac{3}{2}]_q} & 0 & 1 & & \\ & & -\frac{[2]_q}{[\delta_1-\delta_2-\frac{3}{2}]_q} & 0 & 1 & \\ & & & & & 1 \end{pmatrix}. \tag{5.2}$$

$V_{\frac{1}{2}}(\delta_1) \otimes V_{\frac{3}{2}}(\delta_2)$:

$$F_{12}^{-1} = \begin{pmatrix} 1 & & & & & & & \\ & \frac{[\delta_1-\delta_2-1]_q}{[\delta_1-\delta_2-2]_q} & & & & & & \\ & 0 & \frac{[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-2]_q} & & & & & \\ & 0 & 0 & \frac{[\delta_1-\delta_2+1]_q}{[\delta_1-\delta_2-2]_q} & & & & \\ & -\frac{1}{[\delta_1-\delta_2-2]_q} & 0 & 0 & 1 & & & \\ & & -\frac{[2]_q}{[\delta_1-\delta_2-2]_q} & 0 & 0 & 1 & & \\ & & & -\frac{[3]_q}{[\delta_1-\delta_2-2]_q} & 0 & 0 & 1 & \\ & & & & & & & 1 \end{pmatrix}. \tag{5.3}$$

$V_{\frac{1}{2}}(\delta_1) \otimes V_2(\delta_2)$:

$$F_{12}^{-1} = \begin{pmatrix} 1 &&&&&&&&& \\ & \frac{[\delta_1-\delta_2-\frac{3}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q} &&&&&&&& \\ & 0 & \frac{[\delta_1-\delta_2-\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q} &&&&&&& \\ & 0 & 0 & \frac{[\delta_1-\delta_2+\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q} &&&&&& \\ & 0 & 0 & 0 & \frac{[\delta_1-\delta_2+\frac{3}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q} &&&&& \\ & -\frac{1}{[\delta_1-\delta_2-\frac{5}{2}]_q} & 0 & 0 & 0 & 1 &&&& \\ && -\frac{[2]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q} & 0 & 0 & 0 & 1 &&& \\ &&& -\frac{[3]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q} & 0 & 0 & 0 & 1 && \\ &&&& -\frac{[4]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q} & 0 & 0 & 0 & 1 & \\ &&&&&&&&& 1 \end{pmatrix}. \tag{5.4}$$

$V_1(\delta_1) \otimes V_1(\delta_2)$:

$$F_{12}^{-1} = \begin{pmatrix} 1 &&&&&&&& \\ & \frac{[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-2]_q} &&&&&&& \\ & 0 & \frac{[\delta_1-\delta_2]_q[\delta_1-\delta_2+1]_q}{[\delta_1-\delta_2-2]_q[\delta_1-\delta_2-1]_q} &&&&&& \\ & -\frac{[2]_q}{[\delta_1-\delta_2-2]_q} & 0 & 1 &&&&& \\ && -\frac{[2]_q^2[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-2]_q[\delta_1-\delta_2-1]_q} & 0 & \frac{[\delta_1-\delta_2-1]_q}{[\delta_1-\delta_2-2]_q} &&&& \\ && 0 & 0 & 0 & \frac{[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-2]_q} &&& \\ && \frac{[2]_q}{[\delta_1-\delta_2-2]_q[\delta_1-\delta_2-1]_q} & 0 & -\frac{1}{[\delta_1-\delta_2-2]_q} & 0 & 1 && \\ &&&&& -\frac{[2]_q}{[\delta_1-\delta_2-2]_q} & 0 & 1 & \\ &&&&&&&& 1 \end{pmatrix}. \tag{5.5}$$

$V_1(\delta_1) \otimes V_{\frac{3}{2}}(\delta_2)$: the only non-vanishing matrix elements are:

$$F_{12}^{-1}|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle \tag{5.6a}$$

$$F_{12}^{-1}|0\rangle \otimes |1\rangle = \frac{[\delta_1-\delta_2-\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q}|0\rangle \otimes |1\rangle - \frac{[2]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q}|1\rangle \otimes |0\rangle \tag{5.6b}$$

$$\begin{aligned} F_{12}^{-1}|0\rangle \otimes |2\rangle = {} & \frac{[\delta_1-\delta_2-\frac{1}{2}]_q[\delta_1-\delta_2+\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q[\delta_1-\delta_2-\frac{3}{2}]_q}|0\rangle \otimes |2\rangle \\ & - \frac{[2]_q^2[\delta_1-\delta_2-\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q[\delta_1-\delta_2-\frac{3}{2}]_q}|1\rangle \otimes |1\rangle \\ & + \frac{[2]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q[\delta_1-\delta_2-\frac{3}{2}]_q}|2\rangle \otimes |0\rangle \end{aligned} \tag{5.6c}$$

$$\begin{aligned} F_{12}^{-1}|0\rangle \otimes |3\rangle = {} & \frac{[\delta_1-\delta_2+\frac{1}{2}]_q[\delta_1-\delta_2+\frac{3}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q[\delta_1-\delta_2-\frac{3}{2}]_q}|0\rangle \otimes |3\rangle \\ & - \frac{[2]_q[3]_q[\delta_1-\delta_2+\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q[\delta_1-\delta_2-\frac{3}{2}]_q}|1\rangle \otimes |2\rangle \end{aligned}$$

$$+\frac{[2]_q[3]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q[\delta_1-\delta_2-\frac{3}{2}]_q}|2\rangle\otimes|1\rangle \tag{5.6d}$$

$$F_{12}^{-1}|1\rangle\otimes|0\rangle = |1\rangle\otimes|0\rangle \tag{5.6e}$$

$$F_{12}^{-1}|1\rangle\otimes|1\rangle = \frac{[\delta_1-\delta_2-\frac{3}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q}|1\rangle\otimes|1\rangle - \frac{1}{[\delta_1-\delta_2-\frac{5}{2}]_q}|2\rangle\otimes|0\rangle \tag{5.6f}$$

$$F_{12}^{-1}|1\rangle\otimes|2\rangle = \frac{[\delta_1-\delta_2-\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q}|1\rangle\otimes|2\rangle - \frac{[2]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q}|2\rangle\otimes|1\rangle \tag{5.6g}$$

$$F_{12}^{-1}|1\rangle\otimes|3\rangle = \frac{[\delta_1-\delta_2+\frac{1}{2}]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q}|1\rangle\otimes|3\rangle - \frac{[3]_q}{[\delta_1-\delta_2-\frac{5}{2}]_q}|2\rangle\otimes|2\rangle \tag{5.6h}$$

$$F_{12}^{-1}|2\rangle\otimes|0\rangle = |2\rangle\otimes|0\rangle \tag{5.6i}$$

$$F_{12}^{-1}|2\rangle\otimes|1\rangle = |2\rangle\otimes|1\rangle \tag{5.6j}$$

$$F_{12}^{-1}|2\rangle\otimes|2\rangle = |2\rangle\otimes|2\rangle \tag{5.6k}$$

$$F_{12}^{-1}|2\rangle\otimes|3\rangle = |2\rangle\otimes|3\rangle. \tag{5.6l}$$

$V_{\frac{3}{2}}(\delta_1)\otimes V_{\frac{3}{2}}(\delta_2)$: the only non-vanishing matrix elements are:

$$F_{12}^{-1}|0\rangle\otimes|0\rangle = |0\rangle\otimes|0\rangle \tag{5.7a}$$

$$F_{12}^{-1}|0\rangle\otimes|1\rangle = \frac{[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-3]_q}|0\rangle\otimes|1\rangle - \frac{[3]_q}{[\delta_1-\delta_2-3]_q}|1\rangle\otimes|0\rangle \tag{5.7b}$$

$$\begin{aligned}F_{12}^{-1}|0\rangle\otimes|2\rangle &= \frac{[\delta_1-\delta_2]_q[\delta_1-\delta_2+1]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|0\rangle\otimes|2\rangle \\ &\quad -\frac{[2]_q[3]_q[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|1\rangle\otimes|1\rangle \\ &\quad +\frac{[2]_q[3]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|2\rangle\otimes|0\rangle\end{aligned} \tag{5.7c}$$

$$\begin{aligned}F_{12}^{-1}|0\rangle\otimes|3\rangle &= \frac{[\delta_1-\delta_2]_q[\delta_1-\delta_2+1]_q[\delta_1-\delta_2+2]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q[\delta_1-\delta_2-1]_q}|0\rangle\otimes|3\rangle \\ &\quad -\frac{[3]_q^2[\delta_1-\delta_2]_q[\delta_1-\delta_2+1]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q[\delta_1-\delta_2-1]_q}|1\rangle\otimes|2\rangle \\ &\quad +\frac{[3]_q^2[2]_q[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q[\delta_1-\delta_2-1]_q}|2\rangle\otimes|1\rangle \\ &\quad -\frac{[2]_q[3]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q[\delta_1-\delta_2-2]_q}|3\rangle\otimes|0\rangle\end{aligned} \tag{5.7d}$$

$$F_{12}^{-1}|1\rangle\otimes|0\rangle = |1\rangle\otimes|0\rangle \tag{5.7e}$$

$$F_{12}^{-1}|1\rangle\otimes|1\rangle = \frac{[\delta_1-\delta_2-1]_q}{[\delta_1-\delta_2-3]_q}|1\rangle\otimes|1\rangle - \frac{[2]_q}{[\delta_1-\delta_2-3]_q}|2\rangle\otimes|0\rangle \tag{5.7f}$$

$$\begin{aligned}F_{12}^{-1}|1\rangle\otimes|2\rangle &= \frac{[\delta_1-\delta_2-1]_q[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|1\rangle\otimes|2\rangle \\ &\quad -\frac{[2]_q^2[\delta_1-\delta_2-1]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|2\rangle\otimes|1\rangle\end{aligned}$$

$$+\frac{[2]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|3\rangle\otimes|0\rangle \tag{5.7g}$$

$$F_{12}^{-1}|1\rangle\otimes|3\rangle = \frac{[\delta_1-\delta_2]_q[\delta_1-\delta_2+1]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|1\rangle\otimes|3\rangle - \frac{[2]_q[3]_q[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|2\rangle\otimes|2\rangle + \frac{[2]_q[3]_q}{[\delta_1-\delta_2-3]_q[\delta_1-\delta_2-2]_q}|3\rangle\otimes|1\rangle \tag{5.7h}$$

$$F_{12}^{-1}|2\rangle\otimes|0\rangle = |2\rangle\otimes|0\rangle \tag{5.7i}$$

$$F_{12}^{-1}|2\rangle\otimes|1\rangle = \frac{[\delta_1-\delta_2-2]_q}{[\delta_1-\delta_2-3]_q}|2\rangle\otimes|1\rangle - \frac{1}{[\delta_1-\delta_2-3]_q}|3\rangle\otimes|0\rangle \tag{5.7j}$$

$$F_{12}^{-1}|2\rangle\otimes|2\rangle = \frac{[\delta_1-\delta_2-1]_q}{[\delta_1-\delta_2-3]_q}|2\rangle\otimes|2\rangle - \frac{[2]_q}{[\delta_1-\delta_2-3]_q}|3\rangle\otimes|1\rangle \tag{5.7k}$$

$$F_{12}^{-1}|2\rangle\otimes|3\rangle = \frac{[\delta_1-\delta_2]_q}{[\delta_1-\delta_2-3]_q}|2\rangle\otimes|3\rangle - \frac{[3]_q}{[\delta_1-\delta_2-3]_q}|3\rangle\otimes|2\rangle \tag{5.7l}$$

$$F_{12}^{-1}|3\rangle\otimes|0\rangle = |3\rangle\otimes|0\rangle \qquad F_{12}^{-1}|3\rangle\otimes|1\rangle = |3\rangle\otimes|1\rangle \tag{5.7m}$$

$$F_{12}^{-1}|3\rangle\otimes|2\rangle = |3\rangle\otimes|2\rangle \qquad F_{12}^{-1}|3\rangle\otimes|3\rangle = |3\rangle\otimes|3\rangle. \tag{5.7n}$$

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